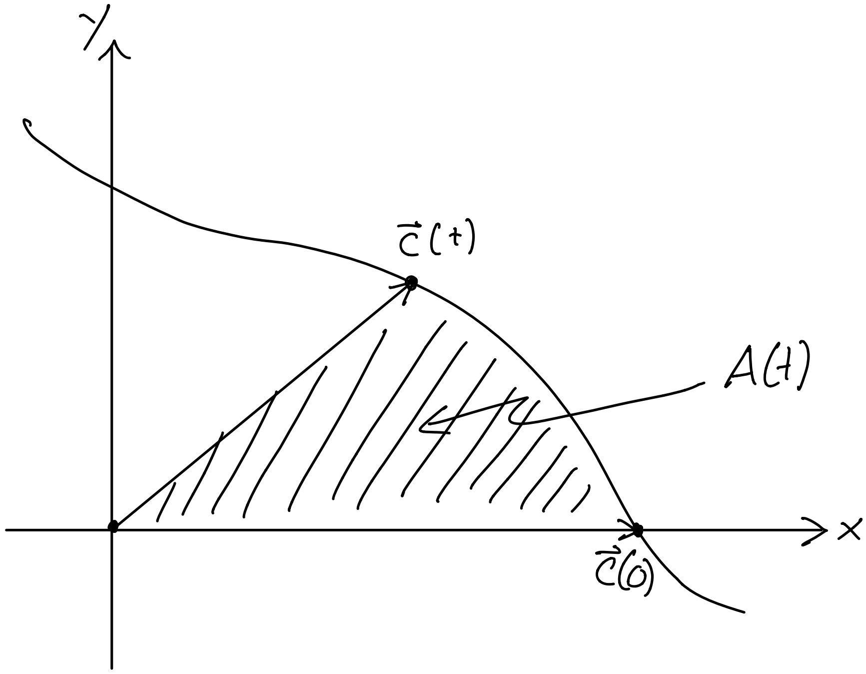
§ 8.4 Areas in Polar Coordinates and planetary motion

Consider an object moving a trajectory $\vec{c}(t)$, where t is time:



We want to compute the area swept out after a time t.

To this end, it is useful to switch to polar coordinates where we have:

$$\vec{c}(t) = r(t)\left(\cos\theta(t), \sin\theta(t)\right)$$

The curve $\vec{c}(t)$, $0 \le t \le T$ is then called a "parametric curve".

We indroduce the notation:

$$\vec{c}(t) = r(t) \cdot \vec{e}(\theta(t)) \qquad (1)$$

where

$$\vec{e}(t) = (\cos t, \sin t)$$

is just the parametrized curve that runs along the unit circle. Note that

$$\vec{e}\,'(t) = (-\sin t, \cos t)$$

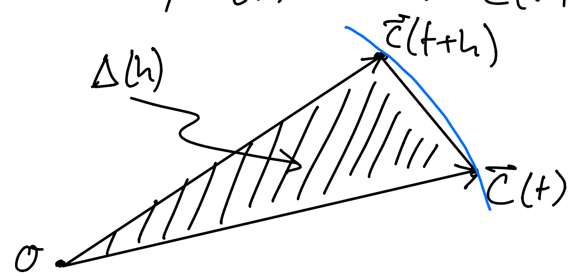
is also a unit vector but perpendicular to $\vec{e}(t)$.

$$\Longrightarrow \det(\vec{e}(t), \vec{e}\,'(t)) = 1 \qquad (2)$$

explicitly:

$$\det\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \cos^2 t - (-\sin^2 t) = 1$$

In order to compute the area $A(t)$, it's useful to first obtain a formula for $A'(t)$. To this end, consider the triangle with vertices 0, $\vec{c}(t)$ and $\vec{c}(t+h)$:



The area of such a triangle can be expressed as:

$$\text{area}(\Delta(t)) = \frac{1}{2} \det\left(\vec{c}(t), \vec{c}(t+h) - \vec{c}(t)\right)$$

Then we compute

$$A'(t) = \lim_{h \to 0} \frac{A(t+h) - A(t)}{h}$$

$$= \lim_{h \to 0} \frac{\text{area}(\Delta(h))}{h}$$

$$= \frac{1}{2} \det\left(\vec{c}(t), \lim_{h \to 0} \frac{\vec{c}(t+h) - \vec{c}(t)}{h}\right)$$

$$= \frac{1}{2} \det\left(\vec{c}(t), \vec{c}'(t)\right)$$

Using

$$\vec{c}'(t) = r'(t) \cdot \vec{e}(\theta(t)) + r(t)\theta'(t)\vec{e}'(\theta(t)) \qquad (3)$$

and linearity of the determinant in its column vectors, we obtain:

$$\det(\vec{c}(t), \vec{c}'(t)) = r(t)r'(t) \overbrace{\det(\vec{e}(\theta(t)), \vec{e}(\theta(t)))}^{=0}$$

$$+ r(t)^2 \theta'(t) \underbrace{\det(\vec{e}(\theta(t)), \vec{e}'(\theta(t)))}_{=1}$$

$$= r^2 \theta' \qquad (4)$$

From this the total area can be obtained by integration:

$$A(t) = \frac{1}{2} \int_0^t r(t)^2 \theta'(t)\, dt$$

or, performing the substitution $d\phi = \theta'(t)dt$:

$$A(\theta) = \frac{1}{2}\int_0^{\theta(t)} \rho(\phi)^2 \, d\phi$$

where $\rho = r \circ \theta^{-1}$. Check:

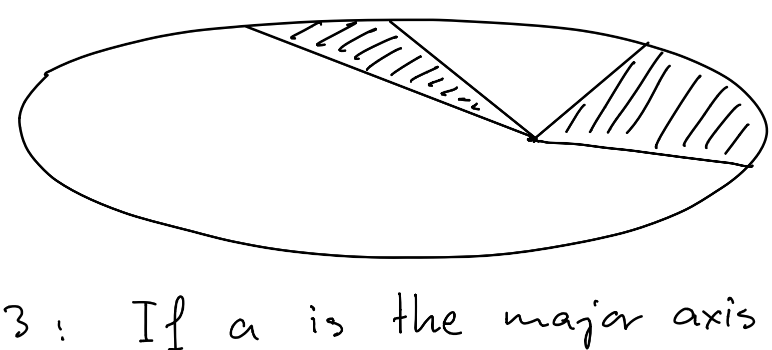
$$A'(t) = \frac{1}{2}\rho(\theta(t))^2 \cdot \theta'(t)$$

$$= \frac{1}{2} r(t)^2 \, \theta'(t) \qquad (5)$$

Let us now use these results to understand Kepler's 3 laws of planetary motion (1609):

K1: The planets move in ellipses, with the sun at one focus.

K2: Equal areas are swept out by the radius vector in equal times.



K3: If a is the major axis of a planet's elliptical orbit and T its period, then a^3/T^2 is the same for all planets.

In our notation, Kepler's second law is equivalent to saying that $A'(t)$ is constant.

so $\quad K2 \iff A'' = 0$

But

$$A'' = \frac{1}{2}\left[\det(\vec{c}, \vec{c}\,')\right]' = \frac{1}{2}\underbrace{\det(c', c')}_{=0} + \frac{1}{2}\det(c, c'')$$

$$= \frac{1}{2}\det(\vec{c}, \vec{c}\,'')$$

(Homework)

So

$$K2 \iff \det(\vec{c}, \vec{c}\,'') = 0$$

From this we can deduce the following:

<u>Proposition 8.2</u> (Newton):

Kepler's second law is true if and only if there exists a "force" that is central, and in this case each planetary path $\vec{c}(t) = r(t) \cdot \vec{e}(\theta(t))$ satisfies the equation

$$r^2 \theta' = \det(\vec{c}, \vec{c}\,') = \text{constant} \qquad (K2)$$

<u>Proof</u>:

Newton introduced the "force" $\vec{F}$ as the product of accelarion and mass: $\vec{F} = m \cdot \vec{c}\,''(t)$

Saying that the force is central just means that it always points along $\vec{c}(t)$. Since $\vec{c}''(t)$ is in the direction of the force, that is equivalent to saying that $\vec{c}''(t)$ always points along $\vec{c}(t)$.

$$\Rightarrow \det(\vec{c}(t), \vec{c}''(t)) = 0.$$

Thus $\det(\vec{c}(t), \vec{c}'(t)) = \text{constant}.$ $\square$

We are now in the position to derive Kepler's first law from Newton's concept of "gravitational force":

Proposition 8.3 (Newton):

If the gravitational force of the sun is a central force that satisfies an "inverse square law", then the path of any planet in it will be an ellipsis having the sun at one focus.

Proof:

By K2 we have $r^2\theta' = \det(\vec{c}, \vec{c}') = M$

for some constant M. The hypothesis of an inverse square law can be written as

$$\vec{c}\,''(t) = -\frac{H}{r(t)^2}\,\vec{e}(\theta(t))$$

for some constant H. Using K2, this can be written as

$$\frac{\vec{c}\,''(t)}{\theta'(t)} = -\frac{H}{M}\,\vec{e}(\theta(t))$$

Notice that the left-hand side of this equation is

$$[\vec{c}\,'(t) \circ \theta^{-1}]'(\theta(t))$$ (use inverse function derivative and chain rule)

So if we let

$$D = \vec{c}\,' \circ \theta^{-1},$$

then the equation can be written as

$$D'(\theta) = -\frac{H}{M}\,\vec{e}(\theta) = -\frac{H}{M}(\cos\theta, \sin\theta),$$

where we now view θ as an indipendent variable. Integrating gives

$$D(\theta) = \left(\frac{H \cdot \sin\theta}{-M} + A,\ \frac{H \cdot \cos\theta}{M} + B\right)$$

for two constants A and B.

Reintroducing the dependence on t, we have

$$\vec{c}\,'(t) = \left(\frac{H \cdot \sin\theta(t)}{-M} + A, \ \frac{H \cdot \cos\theta(t)}{M} + B \right)$$

Substituting this together with $\vec{c} = r(\cos\theta, \sin\theta)$, into the equation

$$\det(\vec{c}, \vec{c}\,') = M,$$

we get

$$r\left[\frac{H}{M}\cos^2\theta + B\cos\theta + \frac{H}{M}\sin^2\theta - A\sin\theta \right] = M,$$

which simplifies to

$$r\left[\frac{H}{M^2} + \frac{B}{M}\cos\theta - \frac{A}{M}\sin\theta \right] = 1.$$

This can be rewritten as (Homework):

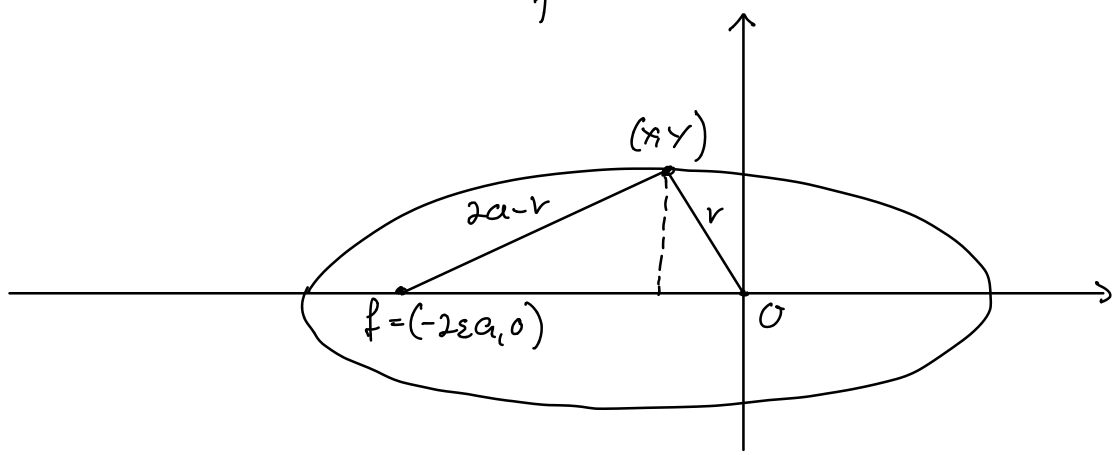
$$r(t)\left[\frac{H}{M^2} + C\cos\left(\theta(t) + D\right) \right] = 1,$$

for some constants C and D. By choosing our polar axis apropriately (which ray corresponds to $\theta = 0$), we can let $D = 0$

$$\Rightarrow \quad r\left[1 + \varepsilon\cos\theta\right] = \frac{M^2}{H} =: \Lambda. \qquad (*)$$

This is the formula for an ellipsis. $\square$

That $(*)$ is the formula for an ellipsis can be seen as follows:



with $0 \le \varepsilon < 1$. We have

$$r^2 = x^2 + y^2 \qquad (1)$$

$$\Rightarrow (2a-r)^2 = (x-(-2\varepsilon a))^2 + y^2,$$

or

$$4a^2 - 4ar + r^2 = x^2 + 4\varepsilon a x + 4\varepsilon^2 a^2 + y^2 \qquad (2)$$

Subtracting (1) by (2) and dividing by $4a$, we get

$$a - r = \varepsilon x + \varepsilon^2 a$$

$$\Leftrightarrow \quad r = a - \varepsilon x - \varepsilon^2 a = (1-\varepsilon^2)a - \varepsilon x$$

$$\Leftrightarrow \quad r = \Lambda - \varepsilon x, \quad \text{for } \Lambda = (1-\varepsilon^2)a.$$

Using $x = r\cos\theta$, we have finally

$$r(1+\varepsilon\cos\theta) = \Lambda.$$